Math for Management, Winter 2023

List 2

Exponents and logarithms

22. Suppose $f(x)$ is a polynomial of degree 5 and $g(x)$ is a polynomial of degree 3. For each of the numbers $0, 1, 2, \dots, 9$, say whether the number could possibly be
    (a) the degree of $f(x) \cdot g(x)$. $\boxed{\text{8 only}}$
    (b) the degree of $f(x) + g(x)$. $\boxed{0, 1, 2, 3, 4, 5}$
    (c) the degree of $f(x) - g(x)$. $\boxed{0, 1, 2, 3, 4, 5}$
    (d) the degree of the remainder when $f(x)$ is divided by $g(x)$. $\boxed{0, 1, 2}$
    (e) the degree of the quotient when $f(x)$ is divided by $g(x)$. $\boxed{\text{2 only}}$

23. Find all the roots of $x^3 - 11x^2 + 29x - 7$ without a calculator. $\boxed{7, 2 - \sqrt{3}, 2 + \sqrt{3}}$

24. Are there non-constant real polynomials $f(x)$ and $g(x)$ such that...
    (a) $fg = x^2 - 25$? $\boxed{\text{Yes}}$: $(x - 5) \cdot (x + 5)$
    (b) $fg = x^2 + 25$? $\boxed{\text{No}}$
    (c) $fg = x^2 - 5$? $\boxed{\text{Yes}}$: $(x - \sqrt{5}) \cdot (x + \sqrt{5})$

25. Which of the following are polynomials?
    (a) $8x^2 + 4x + 1$ $\boxed{\text{polynomial}}$
    (b) $x^{10} + 5x^6 - 100x$ $\boxed{\text{polynomial}}$
    (c) $(x^5 - 2x + 1)(x + 1)$ $\boxed{\text{polynomial}}$
    (d) $(x^5 - 2x + 1)\sin(x)$ $\boxed{\text{not}}$ polynomial
    (e) $3x^2 + 3x^{1/2} - 4$ $\boxed{\text{not}}$ polynomial
    (f) $x^2 + 2^x$ $\boxed{\text{not}}$ polynomial

26. Calculate or simplify the following expressions.
    (a) $\sqrt{32} = \boxed{4\sqrt{2}}$
    (b) $3/\sqrt[3]{9} = \boxed{\sqrt[3]{3}}$
    (c) $\left(\frac{4}{9}\right)^{-1/2} = \boxed{\frac{3}{2}}$
    (d) $8^{5/3} = \boxed{32}$
    (e) $100^{-3/2} = \boxed{\frac{1}{1000}}$
    (f) $4^{-1/4}$ $\boxed{\frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}}$
    (g) $\dfrac{\sqrt[3]{8}}{\sqrt[4]{16}} = \boxed{1}$
    (h) $\sqrt[3]{3 \cdot \sqrt{3}} = \boxed{\sqrt{3}}$

27. Solve the equations:

(a) $4^{2x+1} = 8^{5x-2}$ $\boxed{x = \frac{8}{11}}$

(b) $7 \cdot 3^{x+1} - 5^{x+2} = 3^{x+4} - 5^{x+3}$ $\boxed{x = -1}$

(c) $2^x \cdot 4^{2x} \cdot 8^{3x} = 128$ $\boxed{x = \frac{1}{2}}$

(d) $(3^x)^{2x} \cdot (81^x)^x = 9^{x^2+4}$ $\boxed{x = \sqrt{2}, x = -\sqrt{2}}$

(e) $5^x - 25 \cdot 5^{-x} = 24$ $\boxed{x = 2}$

(f) $\left(\frac{1}{3}\right)^{x-1} = 9^{2x}$ $\boxed{x = \frac{1}{5}}$

"$8 - 2$" is the value of x for which $2 + x = 8$, which is also $x + 2 = 8$.
"$8/2$" is the value of x for which $2 \cdot x = 8$, which is also $x \cdot 2 = 8$.
"$\sqrt[2]{8}$" is the positive value of x for which $x^2 = 8$.
"$\log_2(8)$" is the value of x for which $2^x = 8$.

Writing "$\sqrt{\ }$" without superscript means $\sqrt[2]{\ }$. Depending on context, "log" without any subscript might refer to $\log_{10}$ or $\log_e$ or $\log_2$.

28. Calculate the following (the answers for this task are all rational numbers):

(a) $\log_3(81) = \boxed{4}$

(b) $\log_3 81 = \boxed{4}$

(c) $\log_6(1/36) = \boxed{-2}$

(d) $\log_{81} 3 = \boxed{\frac{1}{4}}$

(e) $\log_{1/2} 4 = \boxed{-2}$

(f) $\log_5 \sqrt{125} = \boxed{\frac{3}{2}}$

(g) $\log_{\sqrt{5}} 125 = \boxed{6}$

(h) $\log_5 9^{\log_3 5} = \boxed{2}$

(i) $\log_6 2 + \log_6 18 = \boxed{2}$

29. Simplify the following into the format $\log_a b$. (Note: $e \approx 2.718...$ is "Euler's constant", and $\ln(x) = \log_e(x)$.)

(a) $\dfrac{\log_{131} 17}{\log_{131} 3} = \boxed{\log_3 17}$

(b) $\log_3 2 - \log_9 2 = \boxed{\log_9 2}$

(c) $\log_{\frac{1}{2}} 3 + \log_4 3 + \log_8 3 = \boxed{\log_{64} \frac{1}{3}}$ This is also $\log_2 \dfrac{1}{\sqrt[6]{3}}$.

30. Simplify $\log_{\frac{1}{2}}\left((e^{\ln 2})^x\right)$. $\boxed{-x}$

31. Find three formulas from the list below that are equal to each other for all $x > 0$.
 $\ln(e^{6+x}) = 6 + x$
 $\boxed{\ln(e^{6x})} = 6x$
 $e^{\ln(6)x} = 6^x$
 $e^{\ln(6+x)} = 6 + x$
 $\boxed{e^{\ln(6x)}} = 6x$
 $\boxed{e^{\ln(6)+\ln(x)}} = 6x$

32. Which number is bigger: $|\log_2 a|$ or $|\log_3 a|$?
 If $0 < a < 1$ then $\log_2 a < \log_3 a < 0$, and $|\log_2 a| > |\log_3 a|$. If $a > 1$ then $\log_2 a > \log_3 > 0$ and so $\boxed{|\log_2 a| > |\log_3 a|}$ again.

33. Find all x such that the expression
$$\frac{1}{2^x + 2^{-x}}$$
 takes values in the interval $(-1, \frac{2}{5})$. $\boxed{x > 1}$

34. What profit will a 1000 zł initial deposit bring after 4 years at an annual interest of 6% if the interest is compounded (also called capitalized) ...
 (a) once a year? There is
$$A = P(1 + \tfrac{r}{n})^{nt} = 1000(1 + 0.06)^4 = 1262.48$$
 at the end of four years, which is a profit of $1262.48 - 1000 = \boxed{262.48 \text{ zł}}$.
 (b) once per month? $A = 1000(1 + \frac{0.06}{12})^{4\cdot 12} = 1270.49$ for a profit of $\boxed{270.49 \text{ zł}}$.
 (c) continuously? $A = Pe^{rt} = 1000e^{0.06\cdot 4} = 1271.25$ for a profit of $\boxed{271.25 \text{ zł}}$.

35. The frequency of the occurrence of the leading (first) digit in many real-life statistical data (from stock prices to the populations of countries) shows a peculiar pattern: the probability of k being the first digit is
$$P_k = \log_{10}\left(\frac{k+1}{k}\right).$$
 This phenomenon, called "Benford's Law", is often used as a way to detect fraud, as most people try to make data sets look random without being aware that some digits occur more frequently than others as leading digits in truly random data.
 (a) Write down frequencies $P_1, P_2, \ldots, P_9$ suggested by Benford's Law.

 $P_1 = \log_{10}(2) \approx 0.693.$
 $P_2 = \log_{10}(3/2) \approx 0.405.$
 $P_3 = \log_{10}(4/3) \approx 0.288.$
 $P_4 = \log_{10}(5/4) \approx 0.223.$
 $P_5 = \log_{10}(6/5) \approx 0.182.$
 $P_6 = \log_{10}(7/6) \approx 0.154.$
 $P_7 = \log_{10}(8/7) \approx 0.134.$
 $P_8 = \log_{10}(9/8) \approx 0.118.$
 $P_9 = \log_{10}(10/9) \approx 0.105.$

(b) Calculate the sum of all the P_k's, and explain the meaning of the obtained value.

You could use the decimals, but an exact answer can be found as

$$\begin{aligned}\log_{10}(2) + \log_{10}(\tfrac{3}{2}) + \log_{10}(\tfrac{4}{3}) + \cdots + \log_{10}(\tfrac{10}{9}) \\ = \log_{10}\left(2 \times \tfrac{3}{2} \times \tfrac{4}{3} \times \cdots \times \tfrac{9}{8} \times \tfrac{10}{9}\right) \\ = \log_{10}(10) = \boxed{1}\end{aligned}$$

In general, probability $p = 1$ means an event must occur.[1] So this calculation confirms that the leading digit must be one of $1, 2, ..., 9$.

(c) Suppose we have a data set of $N = 2000$ numbers. Among those numbers, 452 numbers begin with 2 or 3. Can we claim that this data set satisfies Benford's Law?

We could expect to have $P_2 + P_3 \approx 0.693 = 69.3\%$ of values start with 2 or 3. This would be $2000 \times 0.693 \approx 1386$ numbers. If only 452 numbers start with 2 or 3, this data does not satisfy Benford's Law. (This does not necessarily mean the data have been falsified.)

36. Solve the equations:

(a) $\log_3(x+1) = 2$ $\boxed{x = 8}$

(b) $\log_x 2 - \log_4 x + \frac{7}{6} = 0$ $\boxed{x = 8, x = 2^{-2/3}}$

(c) $\log_2 x + \log_8 x = 12$ $\boxed{x = 512}$

(d) $\log_5 x + \log_5(x+5) = 2 + \log_5 2$ $\boxed{x = 5}$

(e) $(\ln x)^2 + 3\ln x = 4$ $\boxed{x = e, x = e^{-4}}$

37. Solve the inequalities:

(a) $\log_{\frac{1}{2}} x \leq 2$ $\boxed{x \geq \frac{1}{4}}$

(b) $\log_3 x < -\frac{1}{3}$ $\boxed{0 < x < 3^{-1/3}}$

(c) $\log_2 x \geq \log_2(x^2)$ $\boxed{0 < x \leq 1}$

(d) $\log_{\frac{1}{3}} x + 2\log_{\frac{1}{9}}(x-1) \leq \log_{\frac{1}{3}} 6$ $\boxed{x \geq 3}$

(e) $\log_2\left(\log_3\left(\dfrac{x-1}{x+1}\right)\right) > 0$ $\boxed{-2 < x < -1}$

38. Determine the value(s) of m such that the equation

$$x^2 - 2x + \log_{0.5} m = 0$$

has exactly one solution for x. $\boxed{m = \frac{1}{2}}$

39. Solve the system $\begin{cases} x^2 = 9y \\ y - x = -2. \end{cases}$ $\boxed{(x, y) = (3, 1) \text{ or } (x, y) = (6, 4)}$

[1] This assumes there are only finitely many possible outcomes. If there are infinitely many possible outcomes, an event with $p = 1$ might still not occur (and an event with $p = 0$ can occur).

40. Solve the systems of equations:

(a) $\begin{cases} 2\log_3(x) - \log_3(y) = 2 \\ 10^{y-x} = \frac{1}{100} \end{cases}$ $\boxed{(x, y) = (3, 1) \text{ or } (x, y) = (6, 4)}$

(b) $\begin{cases} xy = 36 \\ x^{\log_3 y} = 16 \end{cases}$ $\boxed{(x, y) = (4, 9) \text{ or } (x, y) = (9, 4)}$

(c) $\begin{cases} x^y = 9 \\ y = \log_3(x) + 1 \end{cases}$ $\boxed{(x, y) = (3, 2) \text{ or } (x, y) = (\frac{1}{9}, -1)}$

41. Sketch graphs of the functions:

(a) $y = |3^x|$

(b) $y = |3^x - 3|$

(c) $y = 2^{-x}$

(d) $y = 2^{x+|x|}$

(e) $y = 2^{x^2/|x|}$

(f) $y = \log_3(x - 1)$

(g) $y = \ln |x|$

(h) $y = \log_2(2x)$

(i) $y = \log_x 2$

42. How are the graphs $y = \log_3(x^2)$ and $y = 2\log_3(x)$ different from each other?

If $x > 0$ then $\log_3(x^2) = 2\log_3(x)$. However, x can be negative in $\log_3(x^2)$, while x must be positive for $2\log_3 x$. Therefore the graph $y = \log_3(x^2)$ has two symmetric "copies" of the graph $y = 2\log_3(x)$.

Hints:

??. What can be the degree of the remainder?

27a. Re-write both sides as $2^{\text{something}}$.

33. Is 2^x ever negative? What about 2^{-x}? Also, substitute $y = 2^x$.

34a/b. The final amount is $A = P(1+\frac{r}{n})^{nt}$, where P is the initial ("**p**rincipal") amount, r is the **r**ate, n is the **n**umber of times compounded per year, and t is the number of years ("**t**ime"). **c.** $P = Ae^{rt}$, where t is the number of years.

40a. This is the same as Task 39.

41a. This is the same as $y = 3^x$. **b.** First graph $y = 3^x - 3$. **d/e.** Draw the graph for $x < 0$ and for $x \geq 0$ separately.